(a) $f(0.4)$, (b) $f(0.39)$, (c) $f(1.17)$, (d) $f(1.20)$, (e) $f(1.8)$, (f) $f(1.81)$. Repeat (c)–(d) if the graphing window is zoomed in so that $x = 1.00, 1.01, \ldots, 1.20$ and $y = 1.30, 1.31, \ldots, 1.50$. Repeat (e)–(f) if the graphing window is zoomed in so that $x = 1.800, 1.801, \ldots, 1.820$ and $y = 3.200, 3.205, \ldots, 3.300$.

2. Graph $y = x^2 - 1$, $y = x^2 + x - 1$, $y = x^2 + 2x - 1$, $y = x^2 - x - 1$, $y = x^2 - 2x - 1$ and other functions of the form $y = x^2 + cx - 1$. Describe the effect(s) a change in c has on the graph.

3. Figures 0.31 and 0.32 provide a catalog of the possible types of graphs of cubic polynomials. In this exercise, you will compile a catalog of graphs of fourth-order polynomials (i.e., $y = ax^4 + bx^3 + cx^2 + dx + e$). Start by using your calculator or computer to sketch graphs with different values of a, b, c, d and e. Try $y = x^4$, $y = 2x^4$, $y = -2x^4$, $y = x^4 + x^3$, $y = x^4 + 2x^3$, $y = x^4 - 2x^3$, $y = x^4 + x^2$, $y = x^4 - x^2$, $y = x^4 - 2x^2$, $y = x^4 + x$, $y = x^4 - x$ and so on. Try to determine what effect each constant has.

0.3 INVERSE FUNCTIONS

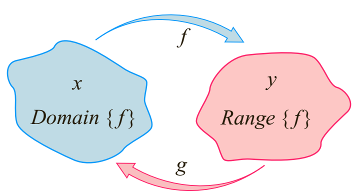


FIGURE 0.41
$g(x) = f^{-1}(x)$

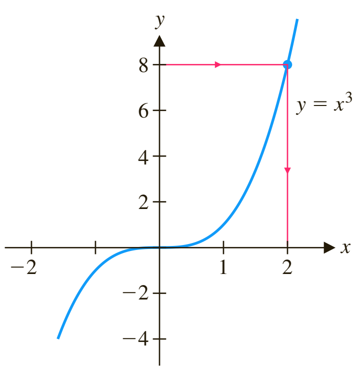


FIGURE 0.42
Finding the x-value corresponding to $y = 8$

The notion of an *inverse* relationship is basic to many areas of science. The number of common inverse problems is immense. As only one example, take the case of the electrocardiogram (EKG). In an EKG, technicians connect a series of electrodes to a patient's chest and use measurements of electrical activity on the surface of the body to infer something about the electrical activity on the surface of the heart. This is referred to as an *inverse* problem, since physicians are attempting to determine what *inputs* (i.e., the electrical activity on the surface of the heart) cause an observed *output* (the measured electrical activity on the surface of the chest).

The mathematical notion of inverse is much the same as that just described. Given an output (in this case, a value in the range of a given function), we wish to find the input (the value in the domain) that produced that output. That is, given a $y \in Range\{f\}$, find the $x \in Domain\{f\}$ for which $y = f(x)$. (See the illustration of the inverse function g shown in Figure 0.41.)

For instance, suppose that $f(x) = x^3$ and $y = 8$. Can you find an x such that $x^3 = 8$? That is, can you find the x-value corresponding to $y = 8$? (See Figure 0.42.) Of course, the solution of this particular equation is $x = \sqrt[3]{8} = 2$. In general, if $x^3 = y$, then $x = \sqrt[3]{y}$. In light of this, we say that the cube root function is the *inverse* of $f(x) = x^3$.

EXAMPLE 3.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x.

Solution For all real numbers x, we have

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

and

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x. \blacksquare$$

Notice in example 3.1 that the action of f undoes the action of g and vice versa. We take this as the definition of an inverse function. (Again, think of Figure 0.41.)

REMARK 3.1

Pay close attention to the notation. Notice that $f^{-1}(x)$ does *not* mean $\dfrac{1}{f(x)}$. We write the reciprocal of $f(x)$ as

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

DEFINITION 3.1

Assume that f and g have domains A and B, respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B \quad \text{and}$$
$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that g is the **inverse** of f, written $g = f^{-1}$. Equivalently, f is the inverse of g, $f = g^{-1}$.

Observe that many familiar functions have no inverse.

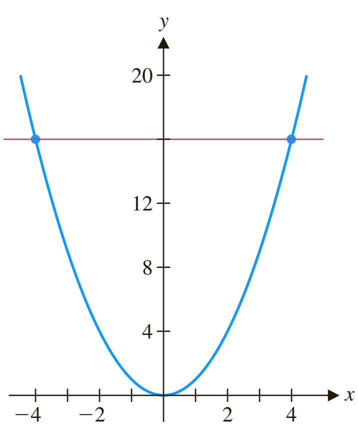


FIGURE 0.43
$y = x^2$

EXAMPLE 3.2 A Function with No Inverse

Show that $f(x) = x^2$ has no inverse on the interval $(-\infty, \infty)$.

Solution Notice that $f(4) = 16$ and $f(-4) = 16$. That is, there are *two* x-values that produce the same y-value. So, if we were to try to define an inverse of f, how would we define $f^{-1}(16)$? Look at the graph of $y = x^2$ (see Figure 0.43) to see what the problem is. For each $y > 0$, there are *two* x-values for which $y = x^2$. Because of this, the function does not have an inverse. ■

For $f(x) = x^2$, it is tempting to jump to the conclusion that $g(x) = \sqrt{x}$ is the inverse of $f(x)$. Notice that although $f(g(x)) = (\sqrt{x})^2 = x$ for all $x \geq 0$ (i.e., for all x in the domain of $g(x)$), it is *not* generally true that $g(f(x)) = \sqrt{x^2} = x$. In fact, this last equality holds *only* for $x \geq 0$. However, for $f(x) = x^2$ restricted to the domain $x \geq 0$, we do have that $f^{-1}(x) = \sqrt{x}$.

DEFINITION 3.2

A function f is called **one-to-one** when for every $y \in \mathit{Range}\{f\}$, there is *exactly one* $x \in \mathit{Domain}\{f\}$ for which $y = f(x)$.

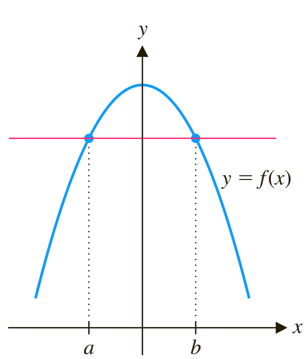


FIGURE 0.44a
$f(a) = f(b)$, for $a \neq b$
So, f does not pass the horizontal line test and is not one-to-one.

REMARK 3.2

Observe that an equivalent definition of one-to-one is the following. A function $f(x)$ is one-to-one if and only if the equality $f(a) = f(b)$ implies $a = b$. This version of the definition is often useful for proofs involving one-to-one functions.

It is helpful to think of the concept of one-to-one in graphical terms. Notice that a function f is one-to-one if and only if every horizontal line intersects the graph in at most one point. This is referred to as the **horizontal line test.** We illustrate this in Figures 0.44a and 0.44b. The following result should now make sense.

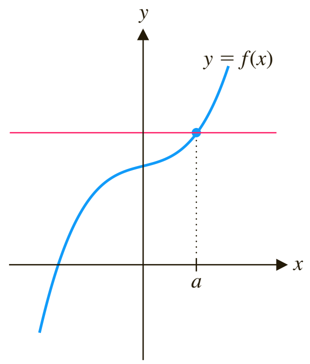


FIGURE 0.44b
Every horizontal line intersects the curve in at most one point. So, f passes the horizontal line test and is one-to-one.

THEOREM 3.1

A function f has an inverse if and only if it is one-to-one.

This theorem simply says that every one-to-one function has an inverse and every function that has an inverse is one-to-one. However, it says nothing about how to find an inverse. For very simple functions, we can find inverses by solving equations.

EXAMPLE 3.3 Finding an Inverse Function

Find the inverse of $f(x) = x^3 - 5$.

Solution Note that it is not entirely clear from the graph (see Figure 0.45) whether f passes the horizontal line test. To find the inverse function, write $y = f(x)$ and solve for x (i.e., solve for the input x that produced the observed output y). We have

$$y = x^3 - 5.$$

Adding 5 to both sides and taking the cube root gives us

$$(y + 5)^{1/3} = (x^3)^{1/3} = x.$$

So, $x = f^{-1}(y) = (y + 5)^{1/3}$. Reversing the variables x and y gives us

$$f^{-1}(x) = (x + 5)^{1/3}. \blacksquare$$

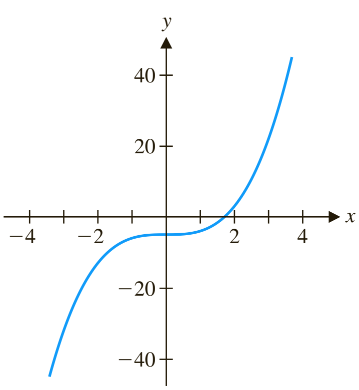


FIGURE 0.45
$y = x^3 - 5$

EXAMPLE 3.4 A Function That Is Not One-to-One

Show that $f(x) = 10 - x^4$ does not have an inverse.

Solution You can see from a graph (see Figure 0.46) that f is not one-to-one; for instance, $f(1) = f(-1) = 9$. Consequently, f does not have an inverse. $\blacksquare$

Most often, we cannot find a formula for an inverse function and must be satisfied with simply knowing that the inverse function exists. Example 3.5 is typical of this situation.

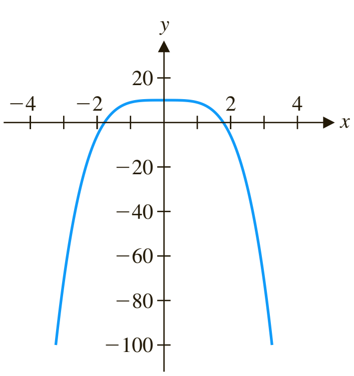


FIGURE 0.46
$y = 10 - x^4$

EXAMPLE 3.5 Finding Values of an Inverse Function

Given that $f(x) = x^5 + 8x^3 + x + 1$ has an inverse, find $f^{-1}(1)$ and $f^{-1}(11)$.

Solution First, notice that from the graph shown in Figure 0.47 (on the following page), the function looks like it might be one-to-one, but how can we be certain of this? (Remember that graphs can be deceptive!) Until we develop some calculus, we will be unable to verify this. Ideally, we would show that f has an inverse by finding a formula for f^{-1}, as in example 3.3. However, in this case, we must solve the equation

$$y = x^5 + 8x^3 + x + 1$$

for x. Think about this for a moment: you should realize that we can't solve for x in terms of y here. We need to assume that the inverse exists, as indicated in the instructions.

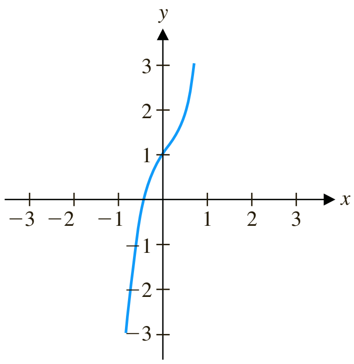


FIGURE 0.47
$y = x^5 + 8x^3 + x + 1$

Turning to the problem of finding $f^{-1}(1)$ and $f^{-1}(11)$, you might wonder if this is possible, since we were unable to find a formula for $f^{-1}(x)$. While it's certainly true that we have no such formula, you might observe that $f(0) = 1$, so that $f^{-1}(1) = 0$. By trial and error, you might also discover that $f(1) = 11$ and so, $f^{-1}(11) = 1$. ■

In example 3.5, we examined a function that has an inverse, although we could not find that inverse algebraically. Even when we can't find an inverse function explicitly, we can say something graphically. Notice that if (a, b) is a point on the graph of $y = f(x)$ and f has an inverse, f^{-1}, then since

$$b = f(a),$$

we have that
$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

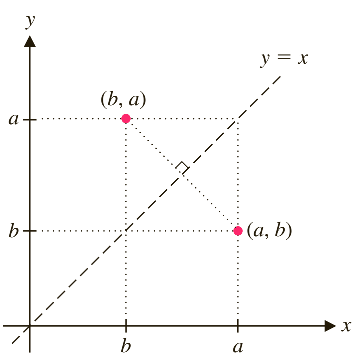


FIGURE 0.48
Reflection through $y = x$

That is, (b, a) is a point on the graph of $y = f^{-1}(x)$. This tells us a great deal about the inverse function. In particular, we can immediately obtain any number of points on the graph of $y = f^{-1}(x)$, simply by inspection. Further, notice that the point (b, a) is the reflection of the point (a, b) through the line $y = x$ (see Figure 0.48). It now follows that given the graph of any one-to-one function, you can draw the graph of its inverse simply by reflecting the entire graph through the line $y = x$.

In example 3.6, we illustrate the symmetry of a function and its inverse.

EXAMPLE 3.6 The Graph of a Function and Its Inverse

Draw a graph of $f(x) = x^3$ and its inverse.

Solution From example 3.1, the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. Notice the symmetry of their graphs shown in Figure 0.49. ■

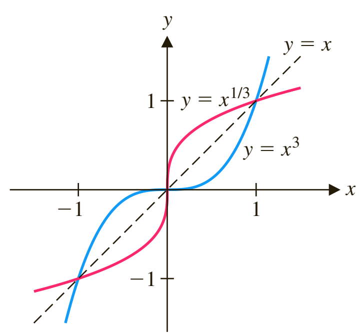


FIGURE 0.49
$y = x^3$ and $y = x^{1/3}$

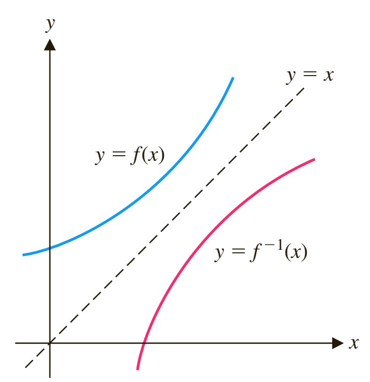


FIGURE 0.50
Graphs of f and f^{-1}

Observe that we can use this symmetry principle to draw the graph of an inverse function, even when we don't have a formula for that function (see Figure 0.50).

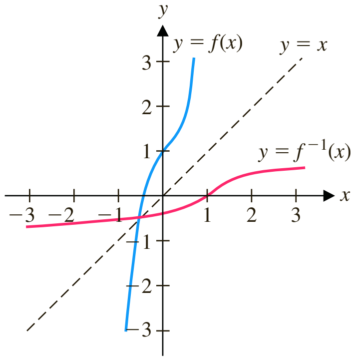


FIGURE 0.51
$y = f(x)$ and $y = f^{-1}(x)$

EXAMPLE 3.7 Drawing the Graph of an Unknown Inverse Function

Draw a graph of $f(x) = x^5 + 8x^3 + x + 1$ and its inverse.

Solution In example 3.5, we were unable to find a formula for the inverse function. Despite this, we can draw a graph of f^{-1} with ease. We simply take the graph of $y = f(x)$ seen in Figure 0.47 and reflect it across the line $y = x$, as shown in Figure 0.51. (When we introduce parametric equations in section 9.1, we will see a clever way to draw this graph with a graphing calculator.) ■

In example 3.8, we apply our theoretical knowledge of inverse functions in a medical setting.

EXAMPLE 3.8 Determining the Proper Dosage of a Drug

Suppose that the injection of a certain drug raises the level of a key hormone in the body. Physicians want to determine the dosage that produces a healthy hormone level. Dosages of 1, 2, 3 and 4 mg produce hormone levels of 12, 20, 40 and 76, respectively. If the desired hormone level is 30, what is the proper dosage?

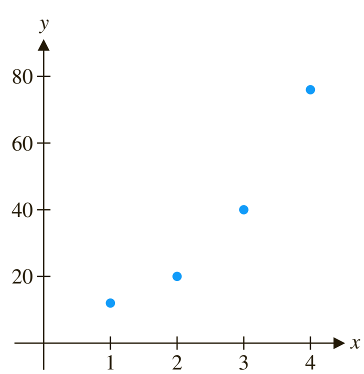


FIGURE 0.52a
Hormone data

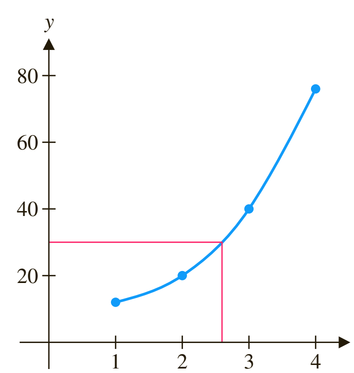


FIGURE 0.52b
Approximate curve

Solution A plot of the points (1, 12), (2, 20), (3, 40) and (4, 76) summarizes the data (see Figure 0.52a). The problem is an inverse problem: given $y = 30$, what is x? It is tempting to argue the following: since 30 is halfway between 20 and 40, the x-value should be halfway between 2 and 3: $x = 2.5$. This method of solution is called **linear interpolation,** since the point $(x, y) = (2.5, 30)$ lies on the line through the points (2, 20) and (3, 40). However, this estimate does not take into account all of the information we have. The points in Figure 0.52a suggest a graph that is curving up. If this is the case, $x = 2.6$ or $x = 2.7$ may be a better estimate of the required dosage. In Figure 0.52b, we have sketched a smooth curve through the data points and indicated a graphical solution of the problem. More advanced techniques (e.g., polynomial interpolation) have been developed by mathematicians to make the estimate of such quantities as accurate as possible. ■

TODAY IN MATHEMATICS

Kim Rossmo (1955–)
A Canadian criminologist who developed the Criminal Geographic Targeting algorithm that indicates the most probable area of residence for serial murderers, rapists and other criminals. Rossmo served 21 years with the Vancouver Police Department. His mentors were Professors Paul and Patricia Brantingham of Simon Fraser University. The Brantinghams developed Crime Pattern Theory, which predicts crime locations from where criminals live, work and play. Rossmo inverted their model and used the crime sites to determine where the criminal most likely lives. The premiere episode of the television drama *Numb3rs* was based on Rossmo's work.

EXERCISES 0.3

WRITING EXERCISES

1. Explain in words (and a picture) why the following is true: if $f(x)$ is increasing for all x, then f has an inverse.
2. Suppose the graph of a function passes the horizontal line test. Explain why you know that the function has an inverse (defined on the range of the function).
3. Radar works by bouncing a high-frequency electromagnetic pulse off of a moving object, then measuring the disturbance in the pulse as it is bounced back. Explain why this is an inverse problem by identifying the input and output.
4. Each human disease has a set of symptoms associated with it. Physicians attempt to solve an inverse problem: given the symptoms, they try to identify the disease causing the symptoms. Explain why this is not a well-defined inverse problem (i.e., logically it is not always possible to correctly identify diseases from symptoms alone).

In exercises 1–4, show that $f(g(x)) = x$ and $g(f(x)) = x$ for all x:

1. $f(x) = x^5$ and $g(x) = x^{1/5}$
2. $f(x) = 4x^3$ and $g(x) = \left(\frac{1}{4}x\right)^{1/3}$
3. $f(x) = 2x^3 + 1$ or $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$
4. $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1-2x}{x}$ $(x \neq 0, x \neq -2)$

In exercises 5–12, determine whether the function is one-to-one. If it is, find the inverse and graph both the function and its inverse.

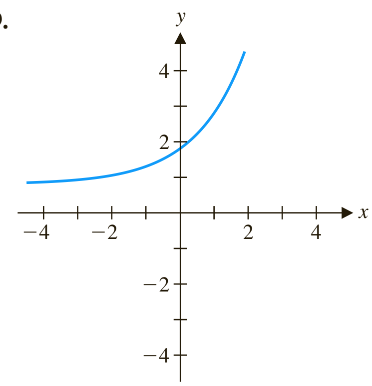
5. $f(x) = x^3 - 2$
6. $f(x) = x^3 + 4$
7. $f(x) = x^5 - 1$
8. $f(x) = x^5 + 4$
9. $f(x) = x^4 + 2$
10. $f(x) = x^4 - 2x - 1$
11. $f(x) = \sqrt{x^3 + 1}$
12. $f(x) = \sqrt{x^2 + 1}$

In exercises 13–18, assume that the function has an inverse. Without solving for the inverse, find the indicated function values.

13. $f(x) = x^3 + 4x - 1$, (a) $f^{-1}(-1)$, (b) $f^{-1}(4)$
14. $f(x) = x^3 + 2x + 1$, (a) $f^{-1}(1)$, (b) $f^{-1}(13)$
15. $f(x) = x^5 + 3x^3 + x$, (a) $f^{-1}(-5)$, (b) $f^{-1}(5)$
16. $f(x) = x^5 + 4x - 2$, (a) $f^{-1}(38)$, (b) $f^{-1}(3)$
17. $f(x) = \sqrt{x^3 + 2x + 4}$, (a) $f^{-1}(4)$, (b) $f^{-1}(2)$
18. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$, (a) $f^{-1}(3)$, (b) $f^{-1}(1)$

In exercises 19–22, use the given graph to graph the inverse function.

19.



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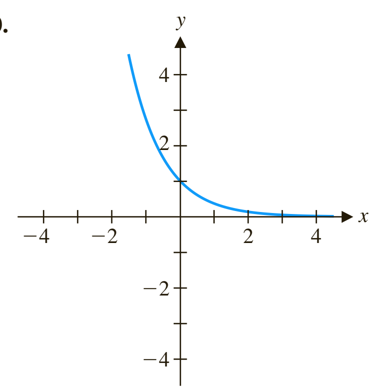

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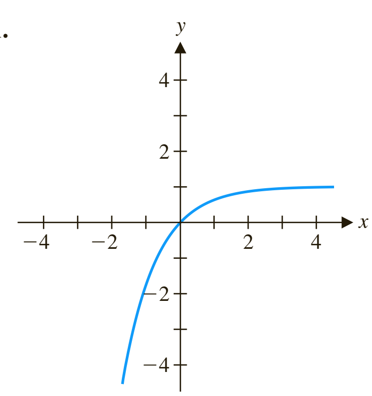
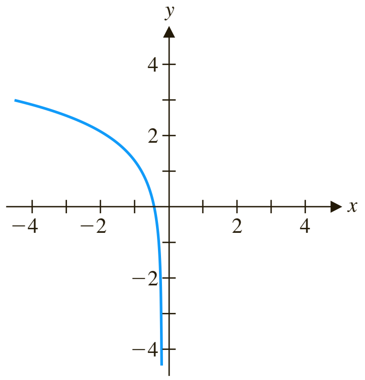


22.



In exercises 23–26, use linear interpolation (see example 3.8) to estimate $f^{-1}(b)$. Use the apparent curving of the graph to conjecture whether the estimate is too high or too low.

23. (1, 12), (2, 20), (3, 26), (4, 30), $b = 23$

24. (1, 12), (2, 10), (3, 6), (4, 0), $b = 8$

25. (1, 12), (2, 6), (3, 2), (4, 0), $b = 5$

26. (1, 12), (2, 20), (3, 36), (4, 50), $b = 32$

In exercises 27–36, use a graph to determine whether the function is one-to-one. If it is, graph the inverse function.

27. $f(x) = x^3 - 5$

28. $f(x) = x^2 - 3$

29. $f(x) = x^3 + 2x - 1$

30. $f(x) = x^3 - 2x - 1$

31. $f(x) = x^5 - 3x^3 - 1$

32. $f(x) = x^5 + 4x^3 - 2$

33. $f(x) = \dfrac{1}{x+1}$

34. $f(x) = \dfrac{4}{x^2+1}$

35. $f(x) = \dfrac{x}{x+4}$

36. $f(x) = \dfrac{x}{\sqrt{x^2+4}}$

Exercises 37–46 involve inverse functions on restricted domains.

37. Show that $f(x) = x^2\ (x \geq 0)$ and $g(x) = \sqrt{x}\ (x \geq 0)$ are inverse functions. Graph both functions.

38. Show that $f(x) = x^2 - 1\ (x \geq 0)$ and $g(x) = \sqrt{x+1}\ (x \geq -1)$ are inverse functions. Graph both functions.

39. Graph $f(x) = x^2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

40. Graph $f(x) = x^2 + 2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

41. Graph $f(x) = (x-2)^2$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

42. Graph $f(x) = (x+1)^4$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

43. Graph $f(x) = \sqrt{x^2 - 2x}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

44. Graph $f(x) = \dfrac{x}{x^2-4}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

45. Graph $f(x) = \sin x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

46. Graph $f(x) = \cos x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

In exercises 47–52, discuss whether the function described has an inverse.

47. The income of a company varies with time.

48. The height of a person varies with time.

49. For a dropped ball, its height varies with time.

50. For a ball thrown upward, its height varies with time.

51. The shadow made by an object depends on its three-dimensional shape.

52. The number of calories burned depends on how fast a person runs.

53. Suppose that your boss informs you that you have been awarded a 10% raise. The next week, your boss announces that due to circumstances beyond her control, all employees will have their salaries cut by 10%. Are you as well off now as you were two weeks ago? Show that increasing by 10% and decreasing by 10% are not inverse processes. Find the inverse for adding 10%. (Hint: To add 10% to a quantity you can multiply the quantity by 1.10.)

EXPLORATORY EXERCISES

1. Find all values of k such that $f(x) = x^3 + kx + 1$ is one-to-one.

2. Find all values of k such that $f(x) = x^3 + 2x^2 + kx - 1$ is one-to-one.